Properties of modules over derived rings.

Recall that for us $A \in CAlg$ is always connective, i.e. $H^i(A) = 0$ for $i \geq 1$.

So the results that I will state will be specialized to this case.

Def'n: A module $M \in Mod_A$ is flat if

- $H^0(M)$ is a ~~$H^0(A)$-flat~~ flat $H^0(A)$-module;
- for all $i \in \mathbb{Z}$,

$$H^i(M) \simeq H^i(A) \underset{H^0(A)}{\otimes} H^0(M).$$

In particular, $M \in Mod_A^{\leq 0}$. And when A is discrete, i.e. $H^i(A) \simeq 0$ $\forall i \neq 0$, then $M \in Mod_A^{\heartsuit}$ and one recovers the usual definition.

Def'n: A module $M \in Mod_A$ is projective if

- $M \in Mod_A^{\leq 0}$
- M is a projective object of $Mod_A^{\leq 0}$, i.e. $Hom_A(M,-)$ commutes w/ geometric realizations.

In particular, A itself is projective.

The conditions above relate to cohomological properties. Let me recap two conditions which concern finiteness, then I will state all the results relating these conditions.

Def'n: A module $M \in Mod_A$ is almost perfect if

- $M \in Mod_A^{\leq k}$ for some $k \in \mathbb{Z}$
- for every $n \geq 0$ $\tau^{\geq -n}(M)$ is compact in $Mod_A^{\geq -n, \leq k}$.

A is almost perfect.

Def'n: A module $M \in Mod_A$ is perfect if

- $Hom(M,-)$ commutes w/ filtered colimits., i.e. M is a compact object of Mod_A.

Here are some equivalent characterizations of projectiveness.

Prop: For a ~~A~~ module $M \in \mathrm{Mod}_A$ ~~is projective~~ TFAE:

(i) M is projective.

(ii) $\forall\, Q \in \mathrm{Mod}_A^{\heartsuit}$ $\mathrm{Ext}^i_A(\overset{M}{\cancel{P}}, Q) = 0$ $\forall\, i > 0$.

(iii) M is a retract of a free A-module, i.e. $\exists$ a commutative diagram

$$\begin{tikzcd}
 & F \arrow[dr, "r"] & \\
M \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_M"'] & & M
\end{tikzcd}.$$

(iv) for any fiber sequence $Q' \to Q \to Q''$ in $\mathrm{Mod}_A^{\le 0}$ $\mathrm{Hom}_{h\mathrm{Mod}_A}(P,Q) \searrow \mathrm{Hom}_{h\mathrm{Mod}_A}(P,Q'')$ is surjective.

Idea of proof: (i) $\Rightarrow$ (ii) $\mathrm{Ext}^i_A(M,Q) = \mathrm{Hom}_{h\mathrm{Mod}_A}(M, Q[i])$.

For $i=1$, let $P_n := \overbrace{0 \underset{Q[i]}{\times} 0 \underset{Q[i]}{\times} \cdots \underset{Q[i]}{\times} 0}^{n+1}$, this gives a simplicial object, such that $|P_\bullet| \simeq Q[i]$. Then

$$\left| \cdots \rightrightarrows \mathrm{Hom}_{\mathrm{Mod}_A}(M, 0 \underset{Q[i]}{\times} 0) \rightrightarrows \mathrm{Hom}_{\mathrm{Mod}_A}(M, 0) \right| \simeq \mathrm{Hom}_{\mathrm{Mod}_A}(M, Q[i]).$$

We are interested in $\pi_0(\mathrm{Hom}_{\mathrm{Mod}_A}(M, Q[i]))$ and for a geometric realization of spaces one has.

$$\pi_0(\mathrm{Hom}_{\mathrm{Mod}_A}(M, Q[i])) = \mathrm{Coker}\left(\pi_0(\mathrm{Hom}_{\mathrm{Mod}_A}(M, 0 \underset{Q[i]}{\times} 0)) \rightrightarrows \pi_0(\mathrm{Mod}_A(M,0))\right)$$

Since, $\pi_0 \mathrm{Hom}_{\mathrm{Mod}_A}(M, 0) \simeq 0$, b/c 0 is a final object, one has $\mathrm{Ext}^1_A(M,Q) \simeq \pi_0(\mathrm{Hom}_{\mathrm{Mod}_A}(M, Q[i])) \simeq 0$.

For $i \geq 2$ one iterates this argument. Also by induction on the truncations

(ii) $\Leftrightarrow$ (ii)' $\forall\, Q \in \mathrm{Mod}_A^{\le 0}$ $\mathrm{Ext}^i_A(M,Q) = 0$ $\forall\, i \geq 1$.

$\Leftrightarrow$ (ii)'' $\forall\, Q \in \mathrm{Mod}_A^{\le 0}$ $\mathrm{Ext}^1_A(M,Q) = 0$.

(ii)'' $\Rightarrow$ (i) is a bit more involved.

factor $\mathrm{Hom}_{\mathrm{Mod}_A}(M,-) : \mathrm{Mod}_A^{\le 0}$

$$\begin{tikzcd}
\mathrm{Mod}_A^{\le 0} \arrow[r, "\mathrm{Hom}_{\mathrm{Mod}_A}(M,-)"] \arrow[dr, dashed, "(ii)''"'] & \mathrm{Spctr} \arrow[r, "\Omega^\infty"] & \mathrm{Spc}. \\
 & \mathrm{Spctr}^{\le 0} \arrow[u, hook] &
\end{tikzcd}$$

Then use **FACT**: [HA, 1.3.3.11]. $\mathcal{C}$, $\mathcal{D}$ stable ∞-cat. w/ left complete t-structures.
$F : \mathcal{C}^{\leq 0} \to \mathcal{D}^{\leq 0}$ is right exact $\Leftrightarrow$ F preserves finite coproducts & 1-1.

Exercise: (iv) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (iii). let $F \xrightarrow{\rho} M$ be a map from a free R-module, surjective on π_0. For instance, take $F := \bigoplus_{x \in \pi_0(M)} R$. Then (iv) $\Rightarrow$ ρ has a section up to homotopy. The result follows from the fact that a retract on $h\mathrm{Mod}_A$ lifts to a retract on Mod_A.

Exercise: (iii) $\Rightarrow$ (i).

Rk: (iv) corresponds to the naive definition one would have made. In particular, when R is discrete it recovers the usual definition. (Check this!)

Let's now discuss flatness. Before that we state a result that is very useful in performing calculations.

~~When M is flat~~

Lemma: Assume M is flat then.

$$H^n(M \otimes_R N) \simeq H^0(M) \otimes_{H^0(R)} H^n(N).$$

The proof uses a spectral seq. argument to compute the LHS.

Here is a characterization of flatness. This sometimes goes by Lazard's theorem.

Prop: For $M \in \mathrm{Mod}_A^{\leq 0}$ TFAE:

(i) M is flat

(ii) M is a filtered colimit of f.g. free A-modules.

(ii)' ______ projective A-modules.

(iii) $M \otimes_R (-) : \mathrm{Mod}_A^{\geq 0} \to \mathrm{Mod}_R^{\geq 0}$

(iii)' $M \otimes_R (-) : \mathrm{Mod}_A^{\heartsuit} \to \mathrm{Mod}_A^{\heartsuit}$.

Idea of proof: (i) $\Rightarrow$ (iii)/(iii)' by the Lemma. (iii) $\Leftrightarrow$ (iii)' Exercise.

(iii)' $\Rightarrow$ (i) that $H^0(M)$ is flat as a $H^0(R)$-module is clear. Use induction on $i \leq 0$. For $i > 0$ it follows from the assumption on M.

((i) ⇒ (ii) is trickier you can read it in [HA. 7.2.2.15].].

(ii) ⇒ (ii)' is tautological.

(ii)' ⇒ (i)

First we argue that any projective M is flat.

Indeed, it is clear that any free module F is flat. Let

$$\begin{tikzcd} & F \arrow[dr, "r"] & \\ M \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_M"'] & & M \end{tikzcd}$$

be a retract. By HTT 4.4.5.18 one has:

$$M \simeq \mathrm{Colim}\,(F \xrightarrow{e} F \xrightarrow{e} F \xrightarrow{e} \cdots) \quad \text{where } e := \mathrm{id}_F - r\circ i.$$

Thus,

$$M \otimes_A N = \mathrm{colim}\,(F \otimes_A N \to F \otimes_A N \to \cdots),$$

if N is discrete, by (iii)' each $F \otimes_A N \in \mathrm{Mod}_A^{\heartsuit}$ is discrete.

Finally, the <u>filtered</u> colimit of discrete objects is discrete. Indeed, as the colimit can be computed in spaces, here it is a homotopy colimit, but since it is filtered is just an ordinary colimit. Then one checks $h^i(-, a)$ commutes w/ filtered colimits for simplicial sets.

The same argument gives that a filtered colimit of projectives is flat. ∎

<u>Rk:</u> Condition (iii) has a generalization. $M \in \mathrm{Mod}_A$ is said to have <u>Tor amplitude $\leq n$</u> if $\forall\, N \in \mathrm{Mod}_A^{\geq 0}$. $M \otimes_A N \in \mathrm{Mod}_A^{\geq -n}$.

Notice: flat = connective + Tor amplitude ≤ 0.

Let's now discuss some equivalent conditions regarding finiteness. For simplicity we will restrict ourselves to the Noetherian case.

<u>Def'n:</u> A derived ring $A \in \mathrm{CAlg}$ is <u>Noetherian</u> if

- $H^0(A)$ is Noetherian.
- for each $i \in \mathbb{Z}$, $H^i(A)$ is "finitely generated. $H^0(A)$-mod (eq. f.p.)

Here is a characterization of perfect modules.

Prop: Let $M \in \mathrm{Mod}_A$ TFAE:

(i) M is perfect

(ii) M ~~belongs to the smallest~~ is a retract of $\bigoplus_{i\in I} R[n_i]$, $|I|<\infty$. for some I. $n_i \in \mathbb{Z}$.

(iii) M is dualizable, i.e. $\exists\, M^\vee$ and maps u, c s.t.

$$M \xrightarrow[\mathrm{id}_M \otimes u]{} M \otimes M^\vee \otimes M \xrightarrow[c \otimes \mathrm{id}_M]{} M \quad \text{is isom. to } \mathrm{id}_M.$$

&

$$M^\vee \xrightarrow[u \otimes \mathrm{id}_{M^\vee}]{} M^\vee \otimes M \otimes M^\vee \xrightarrow[\mathrm{id}_{M^\vee} \otimes c]{} M^\vee \quad \text{is isom. to } \mathrm{id}_{M^\vee}.$$

~~[illegible]~~ (iv) M is almost perfect and has finite Tor amplitude

__Idea of proof:__ (ii) ⇒ (i) R is compact. ~~Filtered colimit of comp~~

FACT: ~~Filtered colimit of~~ Retract of compact is compact.

In ordinary category theory this is clear, since if
$$\begin{tikzcd} M \arrow[r] \arrow[r, bend left] & F \arrow[r] & M \end{tikzcd}$$
is a retract w/ F compact. $M = \mathrm{colim}(F \circlearrowleft F)$ and since finite colimits commute w/ filtered colimits ⇒ $\mathrm{Hom}(M,-)$ preserves colimits.

In ∞-cat. the argument is a bit more subtle the point is.

$$\begin{tikzcd} & \mathrm{Hom}(F,-) \arrow[dr] & \\ \mathrm{Hom}(M,-) \arrow[ur] \arrow[rr] & & \mathrm{Hom}(M,-) \end{tikzcd}$$
is a retract in $\mathrm{Fun}(\mathrm{Mod}_A, \mathrm{Spc})$.

Then $\mathrm{Hom}(F,-)$ compact means for any $Q : K^\triangleright \to \mathrm{Mod}_A$ a colimit, where K is filtered. $\mathrm{Hom}(F, Q\circ(-)) : K^\triangleright \to \mathrm{Fun}(\mathrm{Mod}_A, \mathrm{Spc})$. is a colimit diagram. [HTT 5.1.6.3] ⇒ $\mathrm{Hom}(M, Q\circ(-)) : K^\triangleright \to \mathrm{Fun}(\mathrm{Mod}_A, \mathrm{Spc})$. is also a colimit diagram.

(i) ⇒ (ii) This is equivalent to proving $\mathrm{Ind}(~~[illegible]~~\ \mathrm{Perf}'(R)) \simeq \mathrm{Mod}_R$, where $\mathrm{Perf}(R)' \subseteq \mathrm{Mod}_R$. is the smallest stable ∞-cat. containing R & stable under retracts.

[(ii) ⇒ (i)] ⇒ the inclusion $\mathrm{Perf}(R)' \hookrightarrow \mathrm{Mod}_R$ extends to a fully faithful functor $\mathrm{Ind}(\mathrm{Perf}(R)') \hookrightarrow \mathrm{Mod}_R$. [HTT. 5.3.5.11]

By [HTT. 5.3.5.11] we are left with checking that any $M \in \mathrm{Mod}_R$ is a filtered colimit of objects in $\mathrm{Perf}(R)'$. __Exercise:__ check this!

Hint: $\mathrm{Ind}(\mathrm{Perf}(R)') \to \mathrm{Mod}_R$ preserves all colimits. + ess. image is presentable. use: [HA. Cor. 1.4.4.2]

(i) ⟺ (iii) Notice that (iii) is equivalent to, $\exists M^\vee \in \mathrm{Mod}_A$ s.t.

$$\mathrm{Hom}(M,-) \simeq M^\vee \otimes_A (-) : \mathrm{Mod}_A \to \mathrm{Spc}.$$ are equivalent.

FACT: $\mathrm{Mod}_A \simeq \{ F : \mathrm{Mod}_A \to \mathrm{Spctr} \mid F \text{ is continuous} \}$.
[HA. 7.2.4.3]. $M \mapsto M \otimes_A (-)$.

Notice: $\mathrm{Hom}(M,-)$ automatically commutes w/ direct sums. (i.e. coproducts).

Then $\mathrm{Hom}(M,-)$ is continuous ⟺ M is compact.
FACT ⟹ $\exists M^\vee \in \mathrm{Mod}_A$ w/ $M^\vee \otimes_A (-) \simeq \mathrm{Hom}(M,-)$.

(i) ⟹ (iv) <u>Exercise</u>: M compact in Mod_A ⟹ $\tau^{\geq -n}(M) \in \mathrm{Mod}_A^{\geq -n}$ is compact.

The finite Tor amplitude is an argument using (ii).
Each $\bigoplus_I R[n_i]$ has finite Tor-amplitude.
+ retracts of $\underline{\text{Tor-amplitude}}$ have finite Tor ampl.
The same way that retracts of flat are flat.

(iv) ⟹ (i) is done by induction using the following result.

FACT [HA.7.2.4.20]: M is flat & almost perfect
$\Updownarrow$
M is a retract of a f.g. free R-module.
($\Downarrow$ is clear, $\Uparrow$ uses, + M is projective ⟺ M is flat + $H^0(M)$ is proj. / $H^0(R)$.).
Finally, we mention a cohomological characterization of almost perfect modules.

<u>Prop</u>: For R Noeth. TFAE:
(i) M is almost perfect;

(ii) $H^i(M) = 0$ for $i \gg 0$ & $\forall i \in \mathbb{Z}$, $H^i(M)$ is a f.g. $H^0(M)$-module.

<u>Idea</u>: Induction on $H^i(M)$. For $H^0(M)$ we have. $H^0(M) = \tau^{\geq 0}(M)$ is compact in $\mathrm{Mod}_R^{\heartsuit}$ ⟺ $H^0(M)$ is a f.g. $H^0(R)$-module. since
IS $\underline{\mathrm{Mod}}_{H^0(R)}$ (ord. cat. of $H^0(R)$-modules).

Finally, we mention one last condition which ~~pl~~ plays the role of vector bundles for a derived ring.

Def'n: A module $M \in \mathrm{Mod}_A$ is said to be locally free if M is a retract of a finitely generated free A-module.

One has the following characterization.

Prop: For $M \in \mathrm{Mod}_A$ TFAE

(i) M is locally free;

(ii) M is projective & perfect;

(iii) $\mathrm{Hom}_{\mathrm{Mod}_A^{\leq 0}}(M, -)$ commutes w/ sifted colimits.

(iv) M is dualizable in $\mathrm{Mod}_A^{\leq 0}$.

Idea of proof: (i) $\Leftrightarrow$ (ii) is a consequence of perfect being retracts of ~~⊕~~ ~~⊕~~ finite ~~⊕~~ $\oplus$ sums & shifts of R.

(ii) $\Leftrightarrow$ (iii) Since $\mathrm{Mod}_A^{\leq 0} \hookrightarrow \mathrm{Mod}_A$ commutes w/ colimits we get $\mathrm{Hom}_{\mathrm{Mod}_A^{\leq 0}}(M, -)$ is compact. Then sifted $\simeq$ filt. + geometric realizations.

(i) $\Rightarrow$ (iv) b/c. The collection of dualizable objects is stable under retracts & $\oplus$'s. & A ~~⊗~~ $\in \mathrm{Mod}_A^{\leq 0}$ is dualizable.

(iv) $\Rightarrow$ (i) M is perfect since it is also dualizable in $\mathrm{Mod}A$.

$$H^i(M \otimes_A N) \simeq H^i(\mathrm{Hom}_{\mathrm{Mod}A}(M^\vee, N)). \simeq \mathrm{Hom}_{h\mathrm{Mod}A}(M^\vee[-i], N).$$

Since. $M^\vee \in \mathrm{Mod}_A^{\leq 0}$ for $N \in \mathrm{Mod}_A^{\heartsuit}$ one has. $H^i(M \otimes_A N) = 0$. $\forall i \leq -1$. $\Rightarrow$ M is flat FACT from previous page gives. M is a retract of f.g. free R-module. ◪

~~Notice, when R is discrete. Then~~

We allow ourselves to draw the following representation of the concepts we discussed so far, where the notation should make clear how we defined each category.

$$\begin{tikzcd}
 & \mathrm{Mod}_R \arrow[d, phantom, "\cup"] \arrow[ddl, dashed, "(-)\otimes_R M \text{ is left t-exact.}"'] \arrow[ddr, dashed, "\tau^{\geq -n}(M) \text{ compact } \forall n\geq 0."] & \\
 & \mathrm{Mod}_R^- \arrow[dddr, dashed, "\text{Compact.}"] & \\
\mathrm{Flat}(R) \arrow[ur, bend right] \arrow[d, dashed, bend right, "H^0(M) \text{ projective.}"'] & & \mathrm{APerf}(R) \arrow[ul, bend right] \arrow[d, dashed, bend left, "\text{finite Tor amplitude.}"] \\
\mathrm{Proj}(R) \arrow[u, bend right] \arrow[dr, dashed, "\text{f.g.}"'] & \mathrm{Flat}(R)\cap \mathrm{APerf}(R) \arrow[d, phantom, "\wr"] & \mathrm{Perf}(R) \arrow[u, bend left] \arrow[dl, dashed, "\text{proj.}"] \\
 & \mathrm{Vect}(R) \arrow[ul, bend right] \arrow[ur, bend right] &
\end{tikzcd}$$

In particular, when R is discrete, i.e. $R \simeq H^0(R)$ the above picture simplifies to:

$$\begin{tikzcd}
 & \mathrm{Mod}_{H^0(R)} \arrow[d, phantom, "\cup"] & \\
 & \mathrm{Mod}^-_{H^0(R)} & \\
\mathrm{Flat}(H^0(R)) \arrow[ur, bend right] & & \mathrm{APerf}(H^0(R)) \arrow[ul, bend right] \\
\mathrm{Proj}(H^0(R)) \arrow[u, bend right] & & \mathrm{Perf}(\cancel{R}H^0(R)) \arrow[u, bend left] \\
 & \mathrm{Vect}(\cancel{R}H^0(R)) \arrow[ul, bend right] \arrow[ur, bend right] &
\end{tikzcd}$$

$\mathrm{Flat}(H^0(R)) = \{M \in \mathrm{Mod}^{\heartsuit}_{H^0(R)} \mid M \text{ is flat}\}.$

$\mathrm{APerf}(H^0(R)) \simeq \{M \in \mathrm{Mod}_{H^0(R)} \mid H^i(M) \text{ f.g. } H^0(R)\text{-mod } \forall i \in \mathbb{Z}\}.$

$\mathrm{Perf}(\cancel{R}H^0(R)) \simeq \{M \in \mathrm{Mod}_R^{b} \mid H^i(M) \text{ is f.g. } \forall i \in \mathbb{Z}\} / M$ has finite Tor-amplitude, $M \underset{\text{q.-iso.}}{\simeq} \cdots 0 \to P^i \to \cdots \to P^j \to 0.$ w/ $P^{\bullet}$ projective}

$\mathrm{Proj}(H^0(R)) \simeq \{M \in \mathrm{Mod}^{\heartsuit}_{H^0(R)} \mid M \text{ proj.}\}.$

$\mathrm{Vect}(\cancel{R}H^0(R)) \simeq \{M \in \mathrm{Mod}^{\heartsuit}_{H^0(R)} \mid M \text{ f.g. \& proj.}\}.$

Exercise: Describe all the categories above in the following cases:

(i) $A = k$ a discrete field.

(ii) $A = k[\varepsilon]$ $|\varepsilon| = -1$

(iii) $A = k[y]$ $|y| = -2$.