

Properties of modules over derived rings.

Recall that for us $A \in \text{CAlg}$ is always connective, i.e. $H^i(A) = 0$ for $i \geq 1$.

So the results that I will state will be specialized to this case.

Def'n: A module $M \in \text{Mod}_A$ is flat if

- $H^0(M)$ is a ~~flat~~ flat $H^0(A)$ -module;
- for all $i \in \mathbb{Z}$,

$$H^i(M) \simeq H^i(A) \otimes_{H^0(A)} H^0(M).$$

In particular, $M \in \text{Mod}_A^{\leq 0}$. And when A is discrete, i.e. $H^i(A) \simeq 0 \quad \forall i \neq 0$, then $M \in \text{Mod}_A^{\leq 0}$ and one recovers the usual definition.

Def'n: A module $M \in \text{Mod}_A$ is projective if

- $M \in \text{Mod}_A^{\leq 0}$
- M is a projective object of $\text{Mod}_A^{\leq 0}$, i.e. $\text{Hom}_A(M, -)$ commutes w/ geometric realizations.

In particular, A itself is projective.

~~The~~ The conditions above relate to homological properties. ~~Let~~ Let me recap two conditions which concern finiteness, then I will state all the results relating these conditions.

Def'n: A module $M \in \text{Mod}_A$ is almost perfect if

- $M \in \text{Mod}_A^{\leq k}$ for some ~~large~~ $k \in \mathbb{Z}$
- for every $n \geq 0$ $\tau^{\geq -n}(M)$ is compact in $\text{Mod}_A^{\geq -n, \leq k}$.

A is almost perfect.

Def'n: A module $M \in \text{Mod}_A$ is perfect if

- $\text{Hom}(M, -)$ commutes w/ filtered colimits, i.e. $M \in \text{Mod}_A^{\text{perf}}$
- M is a compact object of Mod_A .

Here are some equivalent characterizations of projectiveness.

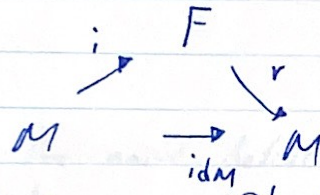
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Prop: A module $M \in \text{Mod}_A$ is projective IFAE:

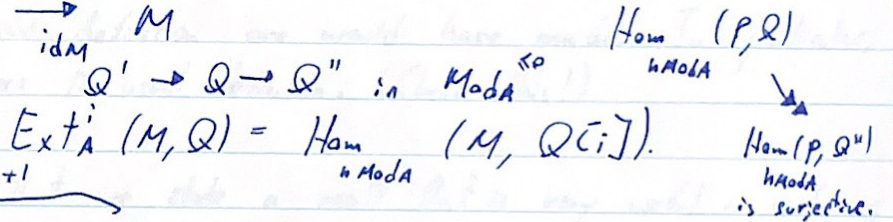
(i) M is projective.

(ii) $\forall Q \in \text{Mod}_A \quad \text{Ext}_A^i(M, Q) = 0 \quad \forall i > 0.$

(iii) M is a retract of a free A -module, i.e. \exists a diagram Commutative.



(iv) for any fiber sequence.



Idea of proof:

(i) \Rightarrow (ii) $\text{Ext}_A^{n+1}(M, Q) = \text{Hom}_{\text{Mod}_A}(M, Q[i])$

For $i=1$, let $P_n := \underbrace{0 \times 0 \times \dots \times 0}_{Q[i] \ Q[i] \ Q[i]}$, this gives a simplicial object, such

that $|P_\bullet| \cong Q[i]$. Then

$$\dots \Rightarrow \text{Hom}_{\text{Mod}_A}(M, 0 \times 0) \Rightarrow \text{Hom}_{\text{Mod}_A}(M, 0) \cong \text{Hom}_{\text{Mod}_A}(M, Q[i]).$$

We are interested in $\text{ho}(\text{Hom}_{\text{Mod}_A}(M, Q[i]))$ and for a geometric realization of spaces one has.

$$\text{ho}(\text{Hom}_{\text{Mod}_A}(M, Q[i])) \xrightarrow{\text{Gker}} \text{ho}(\text{Hom}_{\text{Mod}_A}(M, 0 \times 0)) \cong \text{ho}(\text{Mod}_A(M, 0))$$

Since, $\text{ho} \text{Hom}_{\text{Mod}_A}(M, 0) \cong 0$, b/c 0 is a final object, one

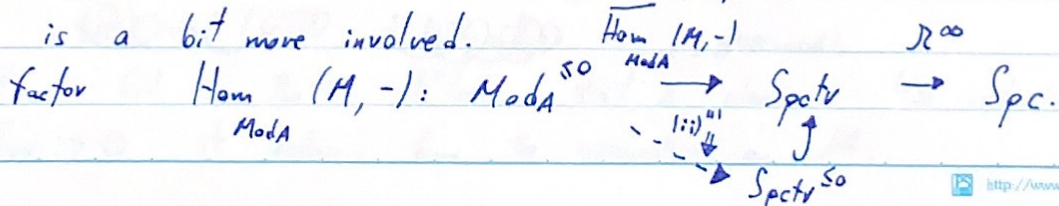
$$\text{has } \text{Ext}_A^1(M, Q) \cong \text{ho}(\text{Hom}_{\text{Mod}_A}(M, Q[i])) \cong 0.$$

For $i \geq 2$ one iterates this argument. Also by induction on the truncations

(ii) \Leftrightarrow (ii)' $\forall Q \in \text{Mod}_A^{\text{fp}} \quad \text{Ext}_A^i(M, Q) = 0 \quad \forall i \geq 1.$

\Leftrightarrow (ii)'' $\forall Q \in \text{Mod}_A^{\text{fp}} \quad \text{Ext}_A^i(M, Q) = 0.$

(ii)'' \Rightarrow (i) is a bit more involved.



Then use [FACT: [COH, 1.3.3-1]]. \mathcal{E}, \mathcal{F} stable ∞ -cat. w/ left complete t -structures.
 $F: \mathcal{E}^{\leq 0} \rightarrow \mathcal{F}^{\leq 0}$ is right exact $\Leftrightarrow F$ ^{preserves} finite coproducts & $|-1$.

\mathcal{E}, \mathcal{F}

Exercise: (iv) \Leftrightarrow (ii).

(i) \Rightarrow (iii). Let $F \xrightarrow{p} M$ be a map from a free R -module, surjective on π_0 , \emptyset . For instance, take $F := \bigoplus R$. Then (iv) $\Rightarrow p$ has a section up to homotopy. The result follows from $\langle \infty, 0 \rangle$ the fact that a retract on $h\text{Mod}_A$ lifts to a retract on Mod_A .

Exercise: (iii) \Rightarrow (i).

RK: (iv) corresponds to the naive definition one would have made. In particular, when R is discrete it recovers the usual definition. (Check this!)

Let's now discuss flatness. Before that we state a result that is very useful in performing calculations.

When M is flat

Lemma: Assume M is flat then.

$$H^n(M \otimes_R N) \cong H^n(M) \otimes_{H^0(R)} H^n(N).$$

\downarrow

The proof uses a spectral seq. argument to compute the LHS.

Here is a characterization of flatness. This sometimes goes by Lazard's theorem.

Prop: For $M \in \text{Mod}_A^{\leq 0}$ TFAE:

(i) M is flat

(ii) M is a filtered colimit of f.g. free A -modules.

(ii)' $\xrightarrow{\quad\quad\quad}$ projective A -modules.

(iii) $M \otimes_R (-) : \text{Mod}_A^{\geq 0} \rightarrow \text{Mod}_R^{\geq 0}$

(iii)' $M \otimes_R (-) : \text{Mod}_A^{\geq 0} \rightarrow \text{Mod}_A^{\geq 0}$.

Idea of proof: (i) \Rightarrow (iii)/(iii)' by the Lemma. (iii) \Leftrightarrow (iii)' Exercise.

(iii) \Rightarrow (ii) that $H^0(M)$ is flat is clear. Use induction on $i \leq 0$.

(iii)' \Rightarrow (i) that $H^0(M)$ is flat is clear. Use induction on $i \leq 0$.

For $i > 0$ it follows from the assumption on M .

(i) \Rightarrow (ii) is trickier you can read it in [HA. 7.2.2.15].

(ii) \Rightarrow (iii)' is tautological.

(iii)' \Rightarrow (i) ~~is~~ ~~trivial~~

First we argue that any projective M is flat.

Indeed, it is clear that any free F is flat. Let

$M \xrightarrow{\text{id}_M} M$ be a retract. By HTT 4.4.5.18 one has:

$$M \cong \text{colim} (F \xrightarrow{e} F \xrightarrow{e} F \xrightarrow{e} \dots) \text{ where } e := \text{id}_F - \text{roi}.$$

Thus, $M \otimes_R N \cong \text{colim} (F \otimes_R N \rightarrow F \otimes_R N \rightarrow \dots)$

if N is discrete, by (iii)' each $F \otimes_R N \in \text{Mod}_A^{\text{discrete}}$ is discrete.

Finally, the filtered colimit of discrete objects is discrete. Indeed, the colimit can be computed in spaces, here it is a homotopy colimit, but since it is filtered is just an ordinary colimit. Then one checks $\mathbb{h}(-, a)$ commutes w/ filtered colimits for simplicial sets.

The same argument gives that a filtered colimit of projectives is flat. \square

Rk: Condition (iii)' has a generalization. $M \in \text{Mod}_A$ is said to have Tor amplitude $\leq n$ if $\forall N \in \text{Mod}_A^{\text{no}}$. $M \otimes_A N \in \text{Mod}_A^{\leq -n}$.

Notice: flat = connective. + Tor amplitude ≤ 0 .

Let's now discuss some equivalent conditions regarding finiteness. For simplicity we will restrict ourselves to the Noetherian case.

Def'n: A derived ring $A \in \text{CAlg}$ is Noetherian if

- $H^0(A)$ is Noetherian.

- for each $i \in \mathbb{Z}$, $H^i(A)$ is "finitely generated". $H^0(A)$ -mod (eq. f.p.)

Here is a characterization of perfect modules

Prop: Let $M \in \text{Mod } A$ TFAE:

- (i) M is perfect
- (ii) M belongs to the smallest \mathcal{C} which is a retract of $\bigoplus_{i \in I} R[u_i]$, $|I| < \infty$, $u_i \in \mathbb{Z}$. for some I .
- (iii) M is dualizable, i.e. $\exists M^\vee$ and maps η, ϵ s.t.

$$M \xrightarrow{\text{id}_M \otimes \eta} M \otimes M^\vee \otimes M \xrightarrow{\epsilon \otimes \text{id}_M} M$$
 is isom. to id_M .
 &

$$M^\vee \xrightarrow{\eta \otimes \text{id}_{M^\vee}} M^\vee \otimes M \otimes M^\vee \xrightarrow{\text{id}_{M^\vee} \otimes \epsilon} M^\vee$$
 is isom. to id_{M^\vee} .

~~Next~~ (iv) M is almost perfect and has finite Tor amplitudes

Idea of proof: (ii) \Rightarrow (i) R is compact. ~~filtered colimit of comp~~

FACT: ~~Filtered colimit of~~ Retract of compact is compact.
 In ordinary category theory this is clear, since if $M \xrightarrow{F} M$ is a retract w/ F compact. $M = \text{colim}(F \circ F)$ and since finite colimits commute w/ filtered colimits $\Rightarrow \text{Hom}(M, -)$ preserves colimits.
 In ∞ -cat. the argument is a bit more subtle the point is:
 $\text{Hom}(M, -) \xrightarrow{\text{Hom}(F, -)} \text{Hom}(M, -)$ is a retract in $\text{Fun}(\text{Mod } A, \text{Spc})$.

Then $\text{Hom}(F, -)$ compact means for any $Q: k^\Delta \rightarrow \text{Mod } A$ a colimit, where k is filtered. $\text{Hom}(F, Q(-)): k^\Delta \rightarrow \text{Fun}(\text{Mod } A, \text{Spc})$ is a colimit diagram. [HTT 5.1.6.3] $\Rightarrow \text{Hom}(M, Q(-)) \xrightarrow{k^\Delta} \text{Fun}(\text{Mod } A, \text{Spc})$ is also a colimit diagram.

(i) \Rightarrow (ii) This is equivalent to proving $\text{Ind}(\text{Perf}^{\text{st}}(R)) = \text{Mod } R$, where $\text{Perf}^{\text{st}}(R) \subseteq \text{Mod } R$ is the smallest stable ∞ -cat. containing R & stable under retracts.

$[(ii) \Rightarrow (i)] \Rightarrow$ the inclusion $\text{Perf}^{\text{st}}(R) \hookrightarrow \text{Mod } R$ extends to a fully

faithful functor $\text{Ind}(\text{Perf}^{\text{st}}(R)) \hookrightarrow \text{Mod } R$.

By [HTT.5.3.5.11] we are left with checking that any $M \in \text{Mod } R$ is a filtered colimit of objects in $\text{Perf}^{\text{st}}(R)$. Exercise: check this!

Hint: $\text{Ind}(\text{Perf}^{\text{st}}(R)) \rightarrow \text{Mod } R$ preserves all colimits. + ess. image is ∞ -presentable. [HA. Gr. 1.4.9.2]

(i) \Leftrightarrow (iii) Notice that (iii) is equivalent to, $\exists M^\vee \in \text{Mod}_A$ s.t.

$\text{Hom}(M, -) \simeq M^\vee \otimes_A (-) : \text{Mod}_A \rightarrow \text{Spce.}$ are equivalent.

FACT: $\text{Mod}_A \simeq \{ F: \text{Mod}_A \rightarrow \text{Spctr} \mid F \text{ is continuous} \}$

[HA. 7.2.4.3]. $M \mapsto M^\vee \otimes_A (-)$.

Notice: $\text{Hom}(M, -)$ automatically commutes w/ direct sums. (i.e. coproducts).

then $\text{Hom}(M, -)$ is continuous $\Leftrightarrow M$ is compact.

FACT $\Rightarrow \exists M^\vee \in \text{Mod}_A$ w/ $M^\vee \otimes_A (-) = \text{Hom}(M, -)$.

$\textcircled{11}$ (i) \Rightarrow (iv) Exercise: M compact in $\text{Mod}_A \Rightarrow \tau^{\geq n}(M) \in \text{Mod}_A$ is compact.

The finite Tor amplitude is an argument using (ii).

Each $\bigoplus_{i=0}^n R\text{Gn}_i$ has finite Tor-amplitude. ~~see (ii) that~~ + retracts of flat have finite Tor-ampl.

The same way that retracts of flat are flat.

(iv) \Rightarrow (i) is done by induction using the following result.

FACT [HA. 7.2.4.20]: M is flat & almost perfect

\Leftrightarrow

M is a retract of a f.g. free R -module.

(\Downarrow is clear, \Uparrow uses \uparrow + M is projective $\Leftrightarrow M$ is flat + $H^0(M)$ is proj. / $H^0(R)$).

Finally, we mention a cohomological characterization of almost perfect modules.

Prop: For R Noeth. TFAE:

(i) M is almost perfect;

(ii) $H^i(M) = 0$ for $i \gg 0$ & $\forall i \in \mathbb{Z}$, $H^i(M)$ is a f.g. $H^0(M)$ -module.

Idea: Induction on $H^i(M)$. For $H^0(M)$ we have. $H^0(M) = \tau^{\geq 0}(M)$

is compact in $\text{Mod}_R \Leftrightarrow H^0(M)$ is a f.g. $H^0(R)$ -module. s.see.

is $\text{Mod}_{H^0(R)}$ (ord. cat. of $H^0(R)$ -modules).

Finally, we mention one last condition which plays the role of vector bundles for a derived ring.

Def'n: A module $M \in \text{Mod}_A$ is said to be locally free if M is a retract of a finitely generated free A -module.

One has the following characterization.

Prop: For $M \in \text{Mod}_A$ TFAE

- (i) M is locally free;
- (ii) M is projective & perfect;
- (iii) $\text{Hom}_{\text{Mod}_A^{\text{so}}}(M, -)$ commutes w/ sifted limits.
- (iv) M is dualizable in Mod_A^{so} .

Idea of proof: (i) \Leftrightarrow (ii) is a consequence of perfect being retracts of finite \oplus sums & shifts of R .

(ii) \Leftrightarrow (iii) Since $\text{Mod}_A^{\text{so}} \hookrightarrow \text{Mod}_A$ commutes w/ colimits we get $\text{Hom}_{\text{Mod}_A^{\text{so}}}(M, -)$ is compact. Then sifted = filt. + geometric realizations.

(i) \Rightarrow (iv) ^{b/c} the collection of dualizable objects is stable under retracts & \oplus 's. & $A \in \text{Mod}_A^{\text{so}}$ is dualizable.

(iv) \Rightarrow (i) M is perfect since it is also dualizable in Mod_A .

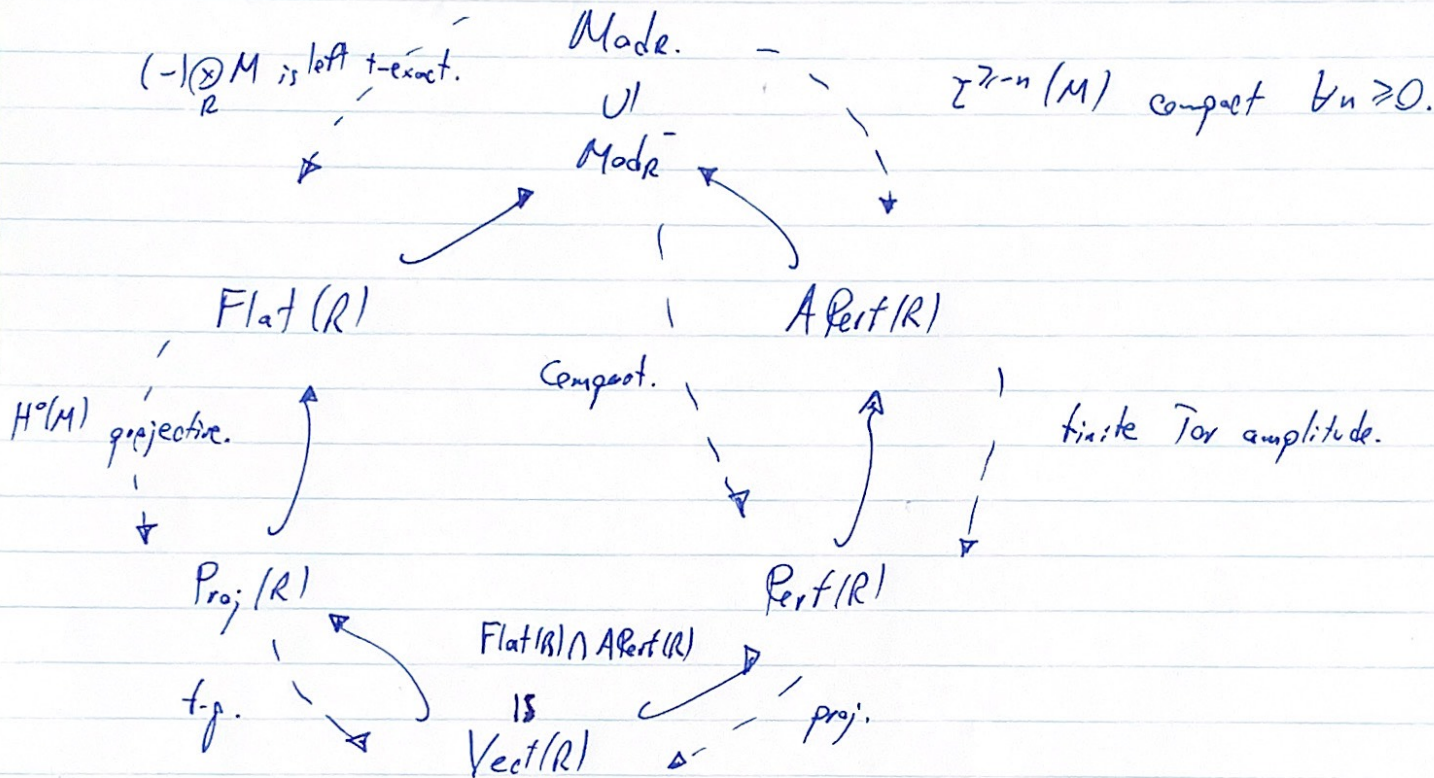
$$H^i(M \otimes_A N) \simeq H^i(\text{Hom}_{\text{Mod}_A}(M^\vee, N)) \simeq \text{Hom}_{\text{Mod}_A}(M^\vee[-i], N).$$

since $M^\vee \in \text{Mod}_A^{\text{so}}$ for $N \in \text{Mod}_A^{\heartsuit}$ one has $H^i(M \otimes_A N) = 0$.

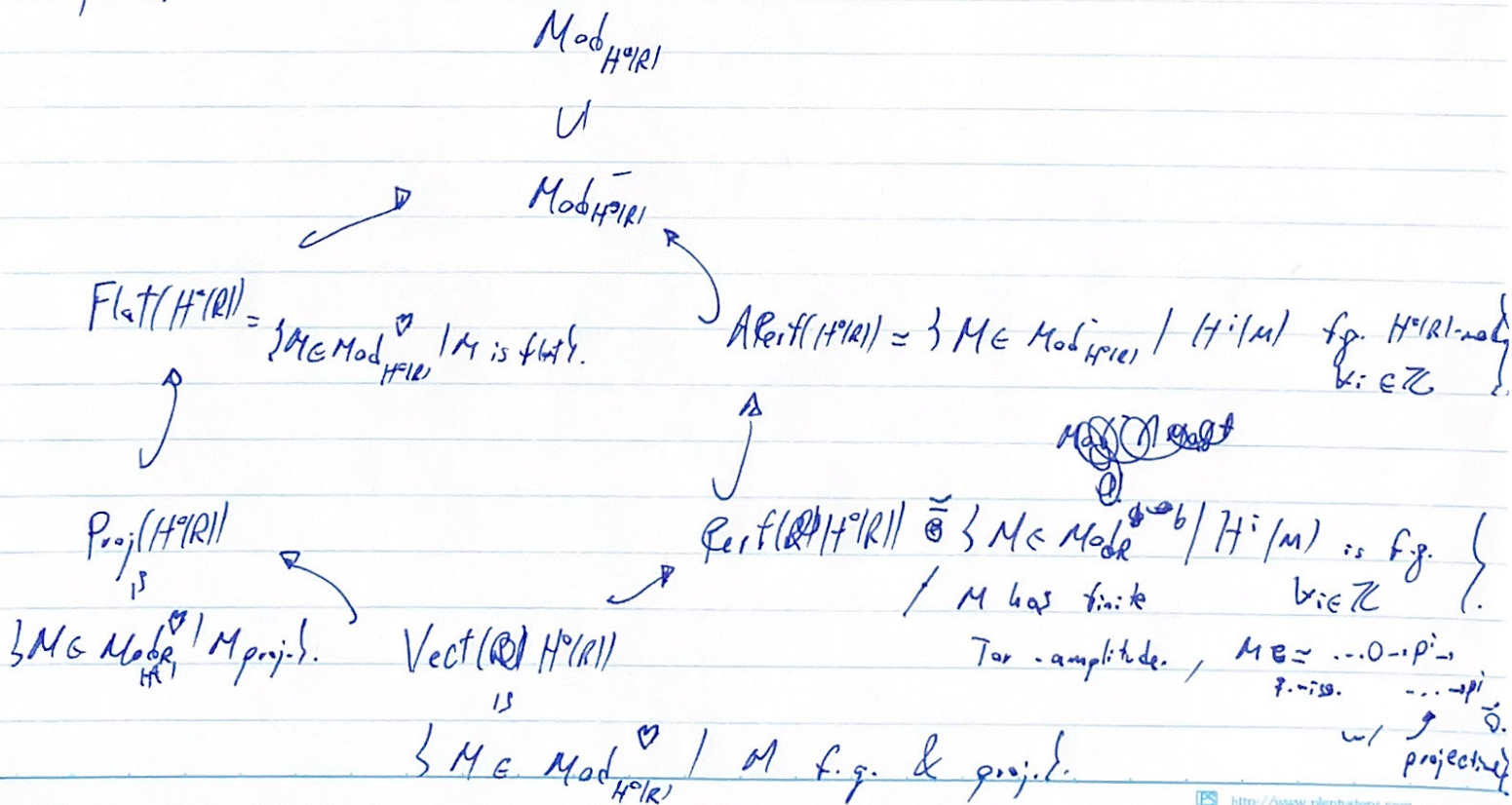
$\forall i \leq -1$. $\Rightarrow M$ is flat FACT from previous page gives.
 M is a retract of f.g. free R -module. \square

Notice when R is discrete. Then

We allow ourselves to draw the following representation of the concepts we discussed so far, where the notation should make clear how we defined each category.



In particular, when R is discrete, i.e. $R \cong H^0(R)$ the above picture simplifies to:



Exercise: Describe all the categories above in the following cases:

(i) $A = k$ a discrete field.

(ii) $A = k[\varepsilon]$ $|\varepsilon| = -1$

(iii) $A = k[y]$ $|y| = -2$.